Parton interactions in the Bjorken asymptotics [1]

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Abstract: We demonstrate the effective action scheme for the leading parton interactions and discuss the symmetry properties. The interaction kernels are particular cases of conformal symmetric two-particle kernels. There is a direct relation to the conformal symmetric rational solution of the Yang-Baxter equation.

1 Introduction

During three decades high energy processes related to the Bjorken limit of scattering amplitudes are playing a major role in the investigation of the hadronic structure and the interaction of hadronic constituents. Quite a number of theoretical concepts and methods have been developed resulting by now in a standard treatment described in textbooks [1] and reviews [2].

In the last years increasing attention is devoted to topics going beyond the standard deep inelastic structure functions, among them the long standing problem of the explicite treatment of higher twist contributions and their scale dependence [3], the generalization to non-forward kinematics [4], interpolating between the scale dependence of structure functions (DGLAP evolution [5]) and the scale dependence of light cone meson wave functions (ERBL evolution [6]) as two limiting cases. The existence of this interpolation has been pointed out early in [7].

The small x behaviour of structure functions and related questions have drawn the attention also to the relations between the Bjorken and the Regge limits. The evolution in the latter asymptotics, as far as it proceeds in the perturbative region, is represented by the BFKL equation [8, 9, 10].

In view of these topics a discussion of the Bjorken asymptotics invoking concepts not aligned to the standard approach may be of interest. We advocate an effective action approach [11] in analogy to the one developed for the Regge asymptotics [12].

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The light ray position as the essential variable of operators in the Bjorken limit appeared explicitely first in the approaches by Geyer, Robaschik et al. [7] and by Balitsky and Braun [14].

The parton interaction operators can be obtained as external field effective vertices, where two type of external fields are introduces describing the asymptotic interaction with the currents of high virtuality and with the hadrons. This calculation can be done analyzing the space-time field configurations only without transforming to momentum space.

From the very beginning conformal symmetry was underlying the ideas about Bjorken scaling and its violation. It turned into a tool for finding multiplicatively renormalized operators [6, 15] and for relating the forward to the non-forward evolutions [13, 16]. Combining conformal and supersymmetry leads to interesting relations between different evolution kernels [13].

Integrability of the effective interaction in high energy QCD amplitudes has been discovered by Lipatov [17] in the Regge limit for the case of multiple exchange of reggeized gluons and a similar symmetry property was expected in the Bjorken limit. The first examples of higher twist evolutions tractable by integrability have been studied by Braun, Korchemsky et al. [18]. Relying on well known methods [19] solutions of the Yang-Baxter equation with conformal ($sl(2)$ and superconformal ($sl(2|1)$) symmetries have been ontained [20] in a form suitable for the Bjorken asmptotics. Some results will be reviewed in sections 3 and 4.

2 Parton interactions in QCD

The standard process is electron - proton scattering $ep \to eX$ at high energy $\sqrt{s}$ with a large momentum transfer $q, -q^2 = xs, x = \mathcal{O}(1)$, from the electron to the hadron. The hadronic part of the process can be written in terms of the forward imaginary part of the virtual photon - proton amplitude.

The non-forward deeply virtual Compton scattering off the proton , $\gamma^*(q_1)+p(p_1) \to \gamma^*(q_2)+p(p_2)$, is an appropriate generalization (Fig. 1a):

$$q_{1/2} = q' - x_{1/2}p', \quad q'^2 = 0, p'^2 = 0, p' \approx p_1, s = 2p'q', \qquad -q_{1/2}^2 = Q_{1/2}^2 = x_{1/2} \;\; s. \tag{2.1}$$

The Bjorken limit corresponds to $s \to \infty$ with the Bjorken variables x_1 and/or x_2 being of order 1.

Let us represent the light cone factorization graphically as in Fig. 1b where a sum over t-channel parton intermediate states is implied. The approach to the asymptotics can be visualized as a process of iterating this factorization. The next step is shown in Fig. 1c.

The Green function H (not necessarily connected) represents the parton interaction. The effective action is to describe just this parton interaction.

In the lowest order of perturbative expansion there are only pair interactions. The partons can be identified as modes of the underlying gluon A and quark $f, \tilde{f}$ fields in the gauge $q'^\mu A_\mu = 0$ after integrating over the redundant field components $p'^\mu A_\mu$ and $p'^\mu \gamma_\mu \; \psi$.

We represent 4-vectors x^μ by their light cone components $x_\pm$ and a complex number involving the transverse components $x_\perp = x_1 + ix_2$. In the case of the gradient vector we change the notation in such a way to have $\partial_+ x_- = \partial_- x_+ = 1, \partial_\perp x = \partial_\perp^* x^* = 1$. In particular the complex valued field A represents the transverse components of the gauge field. We choose the frame where the light-like vector q' has the only non-vanishish component $q'_- = \sqrt{s/2}$ and p' the only non-vanishing component $p'_+ = \sqrt{s/2}$.

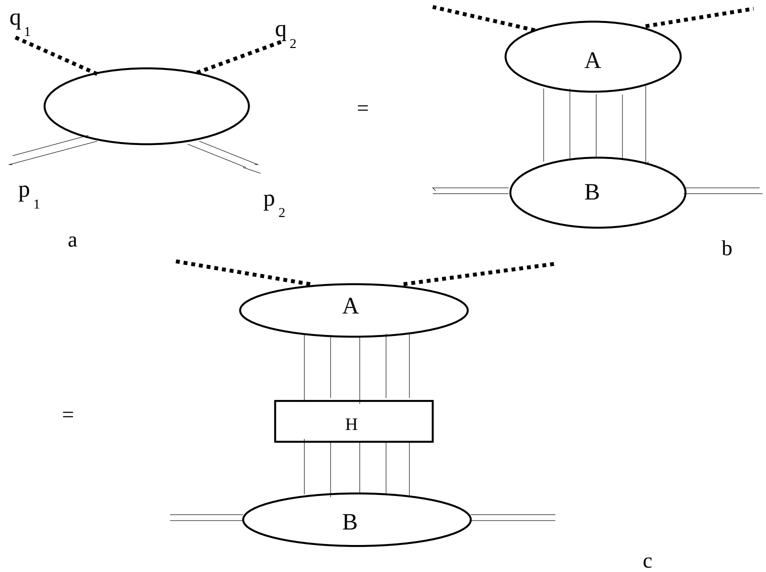

Figure 1: Large-scale factorization of the amplitude. A summation over t-channel intermediate states is implied.

We decompose the Dirac fields into light cone components,

$$\psi = \psi_- + \psi_+, \quad \gamma_-\psi_+ = \gamma_+\psi_- = 0$$
$$\psi_+ = f u_{+-} + \tilde{f} u_{++}, \quad \gamma^\mu = \frac{2}{s}(\gamma_- q'^\mu + \gamma_+ p'^\mu) + \gamma_\perp^\mu. \tag{2.2}$$

$u_{a,b}, \quad a, b = \pm$, is a basis of Majorana spinors,

$$\gamma_+ u_{-,b} = \gamma_- u_{+,b} = 0, \quad \gamma u_{a,-} = \gamma^* u_{a,+} = 0. \tag{2.3}$$

The gauge group structure of the action has been written by using brackets combining two fields into the colour states of the adjoint (a) and of the two fundametal (α and $*\alpha$) representations:

$$(A_1^* T^a A_2) = -i f^{abc} A_1^{*b}\ A_2^c,\ (f_1^* t^a f_2) = t^a_{\alpha\beta} f_1^{*\alpha} f_2^\beta,$$
$$(f^* t^\alpha A) = t^c_{\beta\alpha} f^{*\beta} A^c,\ (A^* t^{*\alpha} f) = t^b_{\alpha\beta} A^{*b} f^\beta. \tag{2.4}$$

We shall restrict the detailed discussion to pure gluodynamics, presenting the final results with fermions included.

The action in light-cone form can be recovered from the kinetic term

$$\int d^4x - 2A^{*a}(\partial_+\partial_- - \partial^\perp\partial^{\perp *})A^a$$
$$= \int d^4x - 2A^{*a}\partial_+\partial_- A^a - \partial^\perp\partial A^{*a}\partial^{-2}\partial^{\perp *}\partial A^a \tag{2.5}$$

by extending the transverse derivatives in the second form of eq. (2.5), relying on the residual gauge symmetry,

$$\partial^\perp A^a \to (\mathcal{D}^\perp A)^a = \partial^\perp A^a + \frac{ig}{2}(A^* T^a A), \tag{2.6}$$

The result can be written as

$$
\begin{aligned}
S \;=\; & \quad \textstyle\int d^4x\{-2A^{a*}(x)(\partial_+\partial_- - \partial_\perp\partial_\perp^*)A^a(x) \\
& \quad +\tfrac{g}{2}(\partial_1\hat{V}^*_{123}[\partial_1 A^a(x_1)(A^*(x_2)T^aA^*(x_3)) + \text{c.c.}) \\
+\frac{g^2}{4} & \quad \hat{V}_{11',22'}[2(A^*(x_1)T^c\partial A(x_{1'})(\partial A^*(x_2)T^cA(x_{2'}))]_{x_i=x_i'=x}\ \}.
\end{aligned}
\tag{2.7}
$$

The elimination of redundant field components has lead to non-local vertices,

$$
\hat{V}^*_{123} = \frac{i}{3\partial_1\partial_2\partial_3}[\partial^*_{\perp 1}(\partial_2 - \partial_3) + \text{cycl.}], \quad \hat{V}_{11',22'} = (\partial_1 + \partial_{1'})^{-2}. \tag{2.8}
$$

Here and in the following we omit the space index $+$ on derivatives, i.e. derivative operators not carrying subscripts $-, \perp$ are to be read as ∂_+. Integer number subscripts refer to the space point on which the derivative acts. The definition of the inverse ∂^{-1} is to be specified.

3 Space-time picture

The block H of the virtual Compton amplitude describes the interaction between sources located in the vicinity of the light ray $x_\perp = 0, x_- = 0, x_+ = z \in \mathbf{R}^1$ and other sources the distribution of which is almost constant in the variables $x'_\perp, x'_+$ depending essentially only on the coordinate along the light-ray $x'_- = z'$. The small width of the first distribution near the light cone is characterized by the short distance scale $\Delta \sim Q^{-1}$ and the variation of the second distribution in directions away from the light ray is determined by the large distance scale $\delta \sim m^{-1}$.

Consider now the QCD functional integral with such sources or the related vertex functional with corresponding external gluon and quark fields. we divide the fields of quarks and gluons into two types of external fields $A^{(\pm)}$ and a quantum fluctuation A_q,

$$
A \to A^{(+)} + A_q + A^{(-)}. \tag{3.1}
$$

$A^{(+)}$ has the support near the light cone and has to be substituted by the following expression in terms of (regularized) delta functions,

$$
A^{(+)}(x) = A_1(z)\delta(x_+)\delta^{(2)}(x_\perp) + \mathcal{O}(\Delta). \tag{3.2}
$$

The other external field $A^{(-)}$ has to be substituted as

$$
A^{(-)}(x') \;=\; A'_1(z') \text{ const} + \mathcal{O}(m). \tag{3.3}
$$

The vertex functional or the external field effective action is now obtained by doing the integration over the quantum fluctuations A_q. Consider in particular the resulting vertex involving two $A^{(+)}$ and two $A^{(-)}$ type fields on the tree level. It has the form

$$
\begin{aligned}
&\int d^4x_1 d^4x_2 d^4x_{1'} d^4x_{'} A^{(+)}(x_1)A^{(+)}(x_2)G(x_{11'})G(x_{22'}) \\
&[\tilde{V}_{11'}G(x_{1'2'})\tilde{V}_{22'} + \delta^{(4)}(x_{1'2'})V_{11',22'}]A^{(-)}(x_{1'})A^{(-)}(x_{2'})
\end{aligned}
\tag{3.4}
$$

Contributions from disconnected graphs have to be added; they result in the cancellation of IR divergencies.

$\tilde{V}_{11'}, \tilde{V}_{22'}$ are simply related to the triple vertex in (2.7) depending on the case considered and $G(x)$ stands for the quark or gluon propagator.

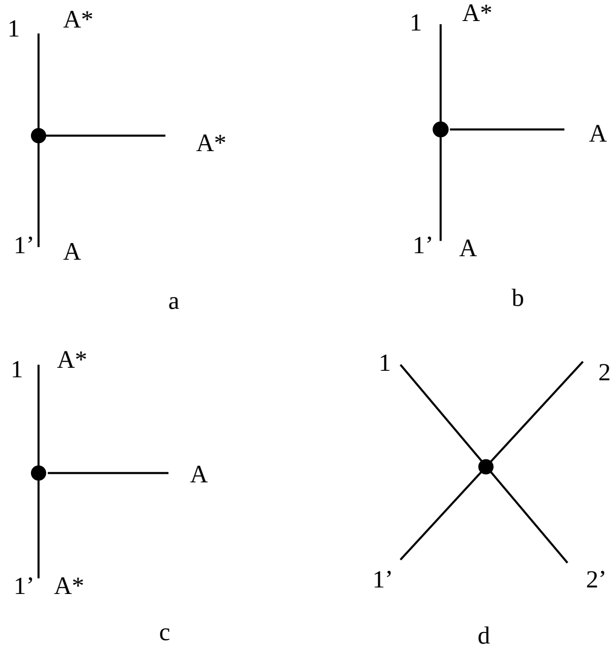

Figure 2: Triple and quartic vertex graphs.

For example, the triple vertex contributes to the tree graph Fig. 2a as

$$-ig \int d^4x_1 x^4 x_{1'} \frac{\partial^{2\prime}}{\partial_1+\partial_{1'}} \left(\frac{\partial_1^{*\perp}}{\partial_1} - \frac{\partial_{1'}^{*\perp}}{\partial_{1'}}\right) A_q^*(A_1^* T^a A_{1'})|_{x_q=x_{1'}} \tag{3.5}$$

and reduces in application to $A^{(+)}(x_1), A^{(-)}(x_{1'})$ to

$$\tilde{V}_{11'} = -ig \int d^4x_1 x^4 x_{1'} \frac{\partial_{1'}^2}{\partial_1+\partial_{1'}} \frac{\partial_1^{*\perp}}{\partial_1} A_q^*(A_1^* T^a A_{1'})|_{x_q=x_{1'}} \tag{3.6}$$

We substitute the particular asymptotic form of the external fields (3.2, 3.3) and obtain

$$c \ln \frac{Q^2}{m^2} \int dz_1 dz_2 dz_{1'} dz_{2'} A_1 A_2 A_{1'} A_{2'} \left[\tilde{V}_{11'}\tilde{V}_{22'} J_\Delta + V_{11',22'} J_0\right] \tag{3.7}$$

We abbreviate the residual dependence of the parton fields on the light ray positions by indices $1,2$ for $A_1^{(+)}(z_1)$, $A_1^{(+)}(z_2)$ and by indices $1',2'$ for $A_1^{(-)\prime}(z_{1'})$, $A_1^{(-)\prime}(z_{2'})$. The integration over the transverse and $+$ components of x_1, x_2 is done due to the delta functions and the integrals over the transverse and $+$ components of $x_{1'}, x_{2'}$ are represented by J_Δ, J_0,

$$\begin{aligned} c \ln \frac{Q^2}{m^2} J_\Delta &= \int dx_{1'+} dx_{2'+} d^2x_{1'\perp} d^2x_{2'\perp} \partial_1^\perp \partial_1^{\perp *} (x_{11'}^2)^{-1} (x_{22'}^2)^{-1} (x_{1'2'}^2)^{-1} \\ c \ln \frac{Q^2}{m^2} J_0 &= \int dx_{1'+} dx_{2'+} d^2x_{1'\perp} d^2x_{2'\perp} \partial_1^\perp \partial_1^{\perp *} (x_{11'}^2)^{-1} (x_{22'}^2)^{-1} \delta^{(4)}(x_{1'2'}) \end{aligned} \tag{3.8}$$

The integrals are regularized by taking into account the smearing of the near light cone distribution by Δ and the large scale cutoff δ for the other distribution. We shall do the integration in the logarithmic approximation. For this we have substituted in (3.7) already the effective triple vertices omitting terms which do not result in logarithmic integrals.

The integral over the transverse coordinates is logarithmic in the region $Q^{-2} < |x'_\perp|^2 < m^{-2}$.

$$c \ln \frac{Q^2}{m^2}\ J_0 = \int \frac{dx'_+ d^2 x'_\perp \delta(z_{1'2'})}{(-x'_+ z_{11'} - |x'_\perp|^2 + i\varepsilon)(-x'_+ z_{22'} - |x'_\perp|^2 + i\varepsilon)} =$$
$$\int_{Q^{-2}}^{m^{-2}} \frac{d^2 x'_\perp}{|x'_\perp|^2} \int_{-\infty}^{\infty} d\bar{\alpha}(\bar{\alpha} z_{11'} + 1 - i\varepsilon)^{-1}(\bar{\alpha} z_{22'} + 1 - i\varepsilon)^{-1} \delta(z_{1'2'}) \tag{3.9}$$

The result can be represented in the convenient form $J_0 = J_0^{(0)}$, where

$$J_0^{(p)}(z_1, z_2; z_{1'}, z_{2'}) = \int_0^1 d\alpha\ \chi_0^{(p)}(\alpha) \delta(z_{11'} - \alpha z_{12})\ \delta(z_{22'} + (1-\alpha) z_{12}), \tag{3.10}$$

$$\chi_0^{(0)} = 1, \chi_0^{(g)} = \alpha(1-\alpha).$$

The case of J_Δ is a bit more involved and can be done as follows.

$$c \ln \frac{Q^2}{m^2} J_\Delta = \int dx_{+1'} x_{+2'} d^2 x_{\perp 1'} d^2_{\perp 2'} (x^\perp_{1'} \cdot x^\perp_{2'})(-x_{+1'} z_{11'} - |x^\perp_{1'}|^2 + i\varepsilon)^{-2}$$
$$(-x_{+2'} z_{22'} - |x^\perp_{2'}|^2 + i\varepsilon)^{-2}(x_{+1'2'} z_{1'2'} - |x^\perp_{1'2'}|^2 + i\varepsilon)^{-1} \tag{3.11}$$

We use the Fock-Schwinger method to do the integrals over the x_+ variables and obtain

$$\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(\sum \alpha_i - 1) \alpha_1 \alpha_2 \delta(\alpha_1 z_{11'} - \alpha_3 z_{1'2'}) \delta(\alpha_2 z_{22'} + \alpha_3 z_{1'2'})$$
$$\int d^2 x_{\perp 1'} d^2_{\perp 2'} (x^\perp_{1'} \cdot x^\perp_{2'})\ (\alpha_1 |x^\perp_{1'}|^2 + \alpha_2 |x^\perp_{2'}|^2 + \alpha_3 |x^\perp_{1'2'}|^2)^{-3}. \tag{3.12}$$

The logarithmic contribution of the transverse integral results in

$$\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(\sum \alpha_i - 1) \frac{\alpha_1 \alpha_2 \alpha_3}{(\alpha_2 + \alpha_3)(\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1)^2}$$
$$\delta(\alpha_1 z_{11'} - \alpha_3 z_{1'2'})\ \delta(\alpha_2 z_{22'} + \alpha_3 z_{1'2'})\ \int_{Q^{-2}}^{m^{-2}} \frac{d|x^\perp_{1'}|^2}{|x^\perp_{1'}|^2}. \tag{3.13}$$

The convenient representation of the result is $J_\Delta = J_{111}$, where

$$J_{n_1 n_2 n_3} = \frac{\Gamma(n_1 + n_2 + n_3 - 1)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(\sum \alpha_i - 1)$$
$$\alpha_1^{n_1 - 1} \alpha_2^{n_2 - 1} \alpha_3^{n_3 - 1}\ \delta(z_{11'} - \alpha_1 z_{12}) \delta(z_{22'} + \alpha_2 z_{12}). \tag{3.14}$$

The general definition of the $\bar{\alpha}$ integrals appearing here is

$$J_{n_1 n_2 \ldots n_N}(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{+\infty} \frac{d\bar{\alpha}}{2\pi i} [\bar{\alpha} x_1 + 1 - i\epsilon]^{-n_1} [\bar{\alpha} x_2 + 1 - i\epsilon]^{-n_2}$$
$$\ldots [\bar{\alpha} x_N + 1 - i\epsilon]^{-n_N}. \tag{3.15}$$

n_i are positive integers. The Feynman parameter representation is

$$J_{n_1 n_2 \ldots}(x_1, x_2, \ldots) = \frac{\Gamma(\sum n_i - 1)}{\Gamma(n_1)\Gamma(n_2) \ldots \Gamma(n_N)} \int \mathcal{D}^{(N)} \alpha\ \alpha_1^{n_1 - 1} \alpha_2^{n_2 - 1}$$
$$\ldots \alpha_N^{n_N - 1}\ \delta(\sum \alpha_i x_i) \tag{3.16}$$

In the latter integral the varibles α_i range from 0 to 1 and $\mathcal{D}^{(N)}\alpha = d\alpha_1...d\alpha_N\delta((1-\sum\alpha_i)$.

By decomposition into simple fractions one can reduce the number of factors in the denominator of (3.15). In this way one derives relations like

$$\begin{aligned} J_{12}(x_1,x_2) &= \frac{x_1}{x_1-x_2}J_{11}(x_1,x_2), \\ J_{111}(x_1,x_2,x_3) &= \frac{x_2}{x_{23}}J_{11}(x_1,x_2) - \frac{x_3}{x_{23}}J_{11}(x_1,x_3), \\ J_{112}(x_1,x_2,x_3) &= \frac{-x_1x_2}{x_{13}x_{23}}J_{11}(x_1,x_2) + \left(\frac{x_2}{x_{23}} + \frac{x_1}{x_{13}}\right) J_{111}(x_1,x_2,x_3), \\ J_{221}(x_1,x_2,x_3) &= \frac{x_3^2}{x_{13}x_{23}}J_{111}(x_1,x_2,x_3) \\ &- \frac{x_1x_2}{x_{12}^2}\left(\frac{x_2}{x_{23}} + \frac{x_1}{x_{13}}\right) J_{11}(x_1,x_2). \end{aligned} \tag{3.17}$$

We consider now in some detail the effective interaction of gluonic partons. The vertex in the external field effective action $A^{*(+)}(x_1)A^{*(+)}(x_2)A^{(-)}(x_{1'})A^{(-)}(x_{2'})$ referring to the case of parallel helicities (chiral odd channel) gets a contribution from the original quartic vertex (2.7) and one from contracting two triple vertices by integrating over the quantum fluctuation A_q. The triple vertices in the Bjorken limit corresponding to Fig. 2a, b are

$$-ig\frac{\partial_{1'}^2\partial_1^{*\perp}}{(\partial_1\partial_{1'})\partial_1}A_q^{a*}\ (A_1^*T^aA_{1'})|_{x_1=x_{1'}=x_q}; \quad ig\frac{\partial_1^2\partial_1^{\perp}}{(\partial_1\partial_{1'})\partial_1}A_q^{a}\ (A_1^*T^aA_{1'})|_{x_1=x_{1'}=x_q} \tag{3.18}$$

Writing down the extenal field vertex as in (3.4), substituting the asymptotic form of the external fields and doing the interations besides of the ones in the light-ray coordinates we obtain as the particular case of (3.7)

$$\int dz_1dz_2dz_{1'}dz_{2'}\left[\frac{\partial_1^2\partial_{2'}^2+\partial_{1'}^2\partial_2^2}{(\partial_1+\partial_{1'})^2\partial_1\partial_2}J_{111} + \frac{\partial_1\partial_{2'}+\partial_{1'}\partial_2}{(\partial_1+\partial_{1'})^2}J_0\right] (A_1^*T^aA_{1'})\ (A_2^*T^aA_{2'}) \tag{3.19}$$

The first term in the brackets is obtained by contracting the triple vertices by substituting the points $1,1'$ in the second and $2,2'$ in the first vertex in (3.18) and vice versa. The second terms emerges from the quartic vertex (2.7) for the case that A_1^*, A_2^* are of $A^{(+)}$ type and the remaining two field of the $A^{(-)}$ type.

The second relation in (3.17) Fourier transformed to light ray variables implies

$$\frac{\partial_1^2\partial_{2'}^2}{(\partial_1+\partial_{1'})^2\partial_1\partial_2}J_{111} \ + \ \frac{\partial_1\partial_{2'}}{(\partial_1+\partial_{1'})^2}J_0 \ = \ \partial_{1'}\partial_{2'}\ \partial_1^{-1}\partial_2^{-1}J_{11'}^{(g)}, \tag{3.20}$$

where

$$J_{11'}^{(p)} = \int_0^1 d\alpha\frac{\chi_1^{(p)}(\alpha)}{\alpha}\ \delta(z_{11'}-\alpha z_{12})\delta(z_{22'}), \tag{3.21}$$
$$\chi_1^{(g)} = -(1-\alpha)^2, \chi_1^{(f)} = -(1-\alpha), \chi_1^{(0)} = \alpha(1-\alpha).$$

We use this relation and the one obtained by interchanging $1,1' \leftrightarrow 2,2'$. The remaining divergence signalled by $\frac{1}{\alpha}$ in the latter integral is cancelled after including the disconnected contribution of order g^2 to the self energy in the two propagators connecting A_1^* with $A_{1'}$ and A_2^* with $A_{2'}$ separately,

$$2\int dz_1dz_2dz_{1'}dz_{2'}\ w_g\ \delta(z_{11'})\delta(z_{22'})\ (A_1^*T^aA_{1'})\ (A_2^*T^aA_{2'}), \tag{3.22}$$

$$w_p = C_p \left(\int_0^1 \frac{d\alpha}{\alpha} \; + \; w_p^{(0)} \right), \quad C_g = N, \;\; C_f = \frac{N^2-1}{2N},$$
$$w_g^{(0)} = -\frac{1}{4}(\frac{11}{3} - \frac{2N_f}{3N}), \quad w_f^{(0)} = -\frac{3}{4}. \tag{3.23}$$

Adding (3.22) to (3.19) the singular part of w_g results in replacing in $J_{11'}^{(g)}$ (3.21) $\frac{1}{\alpha}$ by $\frac{1}{[\alpha]_+}$, adopting the conventional "plus prescription". In the following we understand the factor $\frac{1}{\alpha}$ in $J_{11'}$ with this prescription.

We obtain

$$\int dz_1 dz_2 dz_{1'} dz_{2'} [J_{11'}^{(g)} + w_g^{(0)} \delta^{(2)} + (1,1' \leftrightarrow 2,2')]$$
$$(\partial_1^{-1} A_1^* T^a \partial A_{1'}) \; (\partial_2^{-1} A_2^* T^a \partial A_{2'}), \quad \text{where } \delta^{(2)} \; = \; \delta(z_{11'}) \; \delta(z_{22'}) \tag{3.24}$$

We have calculated in the space-time effective action approach the vertex of parallel helicity gluon interaction to order g^2 in the logarithmic approximation.

In the case of antiparallel helicity gluons the calculation of the effective two-parton interaction results instead of (3.19) in

$$\left(\frac{\partial_{1'}^2 \partial_{2'}^2 + \partial_1^2 \partial_2^2}{(\partial_1 + \partial_{1'})^2 \partial_1 \partial_2} \; J_{111} + \frac{(\partial_{1'} \partial_{2'} + \partial_1 \partial_2)}{(\partial_1 + \partial_{1'})^2} \; J_0 \right) (A_1^* T^a A_{1'}) \; (A_2 T^a A_{2'}^*)$$
$$- \frac{(\partial_1 \partial_{1'} + \partial_2 \partial_{2'})}{(\partial_1 + \partial_2)^2} \; J_0 \;\; (A_1^* T^a A_2) \; (A_{1'} T^a A_{2'}^*)$$
$$+ \frac{(\partial_1 + \partial_{1'})^2}{\partial_1 \partial_2} \; J_{111} \;\; (A_1^* T^a A_{1'}^*) \; (A_2 T^a A_{2'}) \tag{3.25}$$

Here the self-energy contributions are not included yet. We separate in the first bracket a term equal to the result for the parallel helicity case (3.19). Adding now the self-energy contribution (3.22) results in the replacement of the first bracket by

$$2 \partial_{1'} \partial_{2'} [J_{11'} + w_g^{(0)} \delta^{(2)}] - (\partial_{1'} - \partial_1)(\partial_{2'} - \partial_2) \; J_{111} + \partial_1 \partial_2 J_0, \tag{3.26}$$

where now each term represents a regular operator.

The last two relations given in (3.17) for the J integrals Fourier transformed to light ray variables can be used now to substitute $(\partial_{1'} - \partial_1)(\partial_{2'} - \partial_2) \; J_{111}$ and $(\partial_{1'} + \partial_1)^2 \; J_{111}$ by J_{112} and J_{221} plus remainders involving J_0. This results in

$$\left(2[J_{11'} + w_g^{(0)} \delta^{(2)}] + J_{221} - 2J_{112} \right) \; (\partial_1^{-1} A_1^* T^a \partial A_{1'}) \; (\partial_2^{-1} A_2 T^a \partial A_{2'}^*)$$
$$+ J_{221} \;\; (\partial_1^{-1} A_1^* T^a \partial A_{1'}^*) \; (\partial_2^{-1} A_2 T^a \partial A_{2'})$$
$$+ \frac{1}{(\partial_1 + \partial_2)^2} J_0 \;\; \{ -(\partial_1 \partial_{2'} + \partial_2 \partial_{1'}) (A_1^* T^a A_{1'}^*) \; (A_2 T^a A_{2'})$$
$$+ (\partial_1 \partial_{1'} + \partial_2 \partial_{2'}) [(A_1^* T^a A_{1'}) \; (A_2 T^a A_{2'}^*) - (A_1^* T^a A_2) \; (A_{1'} T^a A_{2'}^*)] \}. \tag{3.27}$$

Due to the commutation relation of the generators T^a we have for the gauge group brackets (2.4)

$$(A_1^* T^a A_{1'}) \; (A_2 T^a A_{2'}^*) - (A_1^* T^a A_2) \; (A_{1'} T^a A_{2'}^*) = (A_1^* T^a A_{2'}^*) \; (A_2 T^a A_{1'}). \tag{3.28}$$

Thus the last term in the brackets multiplying J_0 in (3.27) is equal to the first one up to the sign and the exchange of $1'$ and $2'$. We remember that the above expressions are understood as integrated over the light ray positions $1, 2, 1', 2'$. Therefore the exchange of $1'$ and $2'$ is just a substitution of integration variables. We conclude that the contribution involving J_0 cancels and arrive at the result (3.31) up to normalization.

The resulting effective action involves two types of fields $A^{\pm}(z)$ living on the light ray. The (non-local) vertices in this action are converted into Hamiltonians by considering A^- as annihilation and A^+ as creation operators of partons. The Bjorken asymptotics is calculated by studying the evolution generated by those Hamiltonians in the (Euklidian) time variable ξ,

$$\xi(\kappa^2, m^2) = \int_{m^2}^{\kappa^2} \frac{d\kappa'^2}{\kappa'^2} \frac{\alpha_S(\kappa^2)}{2\pi}. \tag{3.29}$$

We list the operators of the leading parton interaction in the QCD Bjorken limit, the momentum kernels of which are well known, in the light-ray and Feynman parameter form. We restrict ourselves to the ones involving gluons (A) and/or only one flavour and one chirality of quarks (f). We use the abbreviations for the kernels (3.10, 3.14, 3.21) and also $J^{\pm} = z_{12}^{\pm 1} J$. The position arguments of the fields will be abbreviated as indices 1, 1′, 2, 2′. We suppress the label ($\pm$) distinguishing creation and annihilation operators, since the fields at points $1, 2$ act always as creation and the fields at the points $1', 2'$ always as annihilation operators. The integration over the positions, summation over colour indices and operator normal ordering is implied.
parallel helicity interactions

$$\begin{aligned} \{4[J^{(g)}_{11'} + w^{(0)}_g \delta^{(2)}](\partial^{-1} A_1^* T^a \partial A_{1'}) - 2i[J^{(f)}_{11'} + w^{(0)}_f \delta^{(2)}](\partial^{-1} f_1^* t^a f_{1'})\} \\ [-2(\partial^{-1} A_2^* T^a \partial A_{2'}) + i(\partial^{-1} f_2^* t^a f_{2'})] \\ -4i J^{(0)}_{11'} (\partial^{-1} f_1^* t^\alpha \partial A_{1'})\ (\partial^{-1} A_2^* t^{*\alpha} f_{2'}) + \text{c.c.} \end{aligned} \tag{3.30}$$

anti-parallel gluon interactions

$$\begin{aligned} \{8[J^{(g)}_{11'} + w^{(0)}_g \delta^{(2)}]\ + 4J_{221} - 8J_{112}\} \\ (\partial^{-1} A_1^* T^a \partial A_{1'})(\partial^{-1} A_2 T^a \partial A_{2'}^*) \\ +4J_{221}(\partial^{-1} A_1^* T^a \partial A_{1'}^*)(\partial^{-1} A_2 T^a \partial A_{2'}) + \text{c.c.} \end{aligned} \tag{3.31}$$

anti-parallel helicity quark interactions (one chirality, one flavour)

$$\begin{aligned} \{-2[J^{(f)}_{11'} + w^{(0)}_f \delta^{(2)}] + J_{111}\}(\partial^{-1} f_1^* t^a f_{1'})(\partial^{-1} f_2 t^a f_{2'}^*) \\ +2J^{(g)}_0 (\partial^{-1} f_1^* t^a f_2)(\partial^{-1} f_{1'} t^a f_{2'}^*) + \text{c.c.} \end{aligned} \tag{3.32}$$

anti parallel helicity quark - gluon interactions

$$\begin{aligned} -4i[J^{(g)}_{11'} + w^{(0)}_g \delta^{(2)}](\partial^{-1} A_1^* T^a \partial A_{1'})\ (\partial^{-1} f_2 t^a f_{2'}^*) \\ -4i[J^{(f)}_{11'} + w^{(0)}_f \delta^{(2)}](\partial^{-1} f_1^* t^a f_{1'})(\partial^{-1} A_2 T^a \partial A_{2'}^*) \\ -4iJ_{211}[(\partial^{-1} f_1^* t^\alpha \partial A_{1'}^*)\ (\partial^{-1} A_2 t^{*\alpha} f_{2'}) - (\partial^{-1} A_1^* T^a \partial A_{1'})(\partial^{-1} f_2 t^a f_{2'}^*)] \\ -8iJ_{111}(\partial^{-1} A_1^* T^a \partial A_{1'})(\partial^{-1} f_2 t^a f_{2'}^*) \quad + \text{c.c.} \end{aligned} \tag{3.33}$$

annihilation-type interactions

$$\begin{aligned} -4iJ^-_{111}\ (\partial^{-1} A_1^* t^\alpha f_{1'}^*)(\partial^{-1} A_2 t^{*\alpha} f_{2'})\ +\ 2i\delta^{(2)-}\ (\partial^{-1} A_1^* t^{*\alpha} f_{1'})(\partial^{-1} A_2 t^\alpha f_{2'}^*) \\ +12iJ_0^{(g)-}\ (\partial^{-1} A_1^* T^a \partial^{-1} A_2)(f_{1'} t^a f_{2'}^*) \\ -\frac{2}{3} iJ^+_{221}\ [(\partial^{-1} f_1^* t^\alpha \partial A_{1'})\ (\partial^{-1} f_2 t^{*\alpha} \partial A_{2'}^*) - (\partial^{-1} f_1^* t^\alpha \partial A_{1'}^*)\ (\partial^{-1} f_2 t^{*\alpha} \partial A_{2'})] \\ -iJ^+_{112}[(\partial^{-1} f_1^* t^\alpha \partial A_{1'})\ (\partial^{-1} f_2 t^{*\alpha} \partial A_{2'}^*) \end{aligned} \tag{3.34}$$

4 Conformal symmetry on the light ray

The conformal transformations act on the light ray positions z as

$$z \to z' = \frac{az+b}{cz+d}$$

This action is generated by

$$S^{(0),-} = \partial, \quad S^{(0)0} = z\partial, \quad S^{(0)+} = -z^2\partial. \tag{4.1}$$

The generators obey the $sl(2)$ algebra,

$$[S^0, S^{\pm}] = \pm S^{\pm}, \quad [S^+, S^-] = 2S^0 \tag{4.2}$$

We need other representations ℓ on functions of z generated by

$$S^{(\ell)-} = S^{(0)-}, \quad S^{(\ell)0} = z^{-\ell} S^{(0)0} z^{\ell}, \quad S^{(\ell)+} = z^{-2\ell} S^{(0)+} z^{2\ell}. \tag{4.3}$$

The representation space $V^{(\ell)}$ is spanned by z^m, and the function 1 represents the lowest weight state.

The tensor product representation, $V^{(\ell_1)} \otimes V^{(\ell_2)}$, is represented by polynomial functions of two variables $\psi(z_1, z_2)$ and is generated by $S^a_{12} = S_1^{(\ell_1)a} + S^{(\ell_2)a}$. It decomposes into irreducible representations, the lowest weight states of which, $S^-_{12}\psi = 0$, $S^0_{12}\psi = \lambda\psi$, are $\psi^{(0)}_n = (z_1 - z_2)^n$, with $\lambda = \ell_1 + \ell_2 + n$.

The two-parton Hamiltonians obtained from QCD can be considered as acting on the tesor product representation spaces

$$\hat{H} : V^{(\ell_{1'})} \otimes V^{(\ell_{2'})} \to V^{(\ell_1)} \otimes V^{(\ell_2)}$$

The operator is symmetric if

$$\left(S_1^{(\ell_1)a} + S_2^{(\ell_2)a} \right) \hat{H} = \hat{H} \left(S_1^{(\ell_{1'})a} + S_2^{(\ell_{2'})a} \right). \tag{4.4}$$

Writing the operator in integral form,

$$\hat{H}\psi(z_1, z_2) = \int dz_{1'} dz_{2'} \; J(z_1, z_2 | z_{1'}, z_{2'}) \psi(z_{1'}, z_{2'}), \tag{4.5}$$

this results in the following condition on the kernel,

$$\left(S_1^{(\ell_1 a} + S_2^{(\ell_2)a} + S_{1'}^{(1-\ell_{1'})a} + S_{2'}^{(1-\ell_{2'})a} \right) J(z_1, z_2 | z_{1'}, z_{2'}) = 0. \tag{4.6}$$

Here we assume that the integration is over closed contours in order to have no boundary terms in the interation by parts. The condition on the kernel is solved by the simple expression

$$J_{\ell_1, \ell_2, \ell_{1'}, \ell_{2'}}(z_1, z_2 | z_{1'}, z_{2'}) = z_{12}^{a_{12}} \; z_{1'2'}^{a_{1'2'}} \; z_{11'}^{a_{11'}} \; z_{22'}^{a_{22'}} \, z_{12'}^{a_{12'}} \; z_{1'2}^{a_{1'2}}, \tag{4.7}$$

where the exponents are given in terms of the conformal weights ℓ_i and two paramenters d, M as

$$a_{12} = d + 1 - \frac{1}{2}\sum \ell_i, \quad a_{1'2'} = d - 1 + \frac{1}{2}\sum \ell_i$$

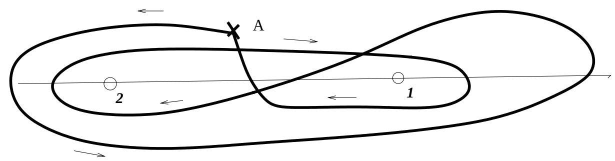

Figure 3: Double-loop contour.

$$
\begin{aligned}
a_{12'} = -\frac{1}{2}(\ell_1+\ell_{1'}-\ell_2-\ell_{2'})+M, \quad a_{1'2} = +\frac{1}{2}((\ell_1+\ell_{1'}-\ell_2-\ell_{2'})+M \\
a_{11'} = -(\ell_1-\ell_{1'}+1)-d-M, \quad a_{22'} = (\ell_1-\ell_{1'}+1)-d-M \\
\sigma = \ell_1-\ell_{1'}+\ell_2-\ell_{2'}
\end{aligned} \tag{4.8}
$$

The general solution is a superposition of the above expression with varying paramenters M and d. In order to close the contour one should choose a double-loop enclosing two branch points as in Fig. 2 (Pochhammer contour). For our applications we need integral operators where the integration is along the real axis in z'_1 and z'_2.

The kernel for closed contours transforms into the kernel for integration along the light ray by contraction of the contours to the branch cuts and rewriting the integral over finite intervals in α representation.

In the regular case, if the exponents are larger than -1, we obtain the kernel in the form

$$
\begin{aligned}
z_{12}^{\sigma}\int_0^1 d\alpha_1 \int_0^{\alpha_1} d\alpha_2\ \delta(z_{11'}-\alpha_1 z_{12})\delta(z_{22'}+\alpha_2 z_{12}) \\
\alpha_1^{a_{11'}}\alpha_2^{a_{22'}}(1-\alpha_1)^{a_{21'}}(1-\alpha_2)^{a_{12'}}(1-\alpha_1-\alpha_2)^{a_{1'2'}}
\end{aligned} \tag{4.9}
$$

Singular cases arise if some powers approach negative integers, e.g. if $a=-1+\varepsilon$ then the kernel is obtained from the one above by by the formal substitution

$$
\frac{1}{\alpha^{1-\varepsilon}} \to \delta(\alpha)+\varepsilon\frac{1}{[\alpha]_+}. \tag{4.10}
$$

Now it is not difficult to check that the kernels obtained in QCD have the conformally symmetric form with the weights $\ell=\frac{3}{2}$ for gluons and $\ell=1$ for quarks.

For example if $\ell_1=\ell_2=\ell_{1'}=\ell_{2'}=\ell$, $M=0, d=\varepsilon$ we have in the integrand (4.9)

$$
\frac{(1-\alpha_1-\alpha_2)^{\varepsilon+2\ell-1}}{\alpha_1^{1-\varepsilon}\alpha_2^{1-\varepsilon}} \to \delta(\alpha_1)\delta(\alpha_2)+\varepsilon\left(\delta(\alpha_1)\frac{(1-\alpha_2)^{2\ell-1}}{[\alpha_2]_+}+\delta(\alpha_2)\frac{(1-\alpha_1)^{2\ell-1}}{[\alpha_1]_+}\right) \tag{4.11}
$$

The second term results in the QCD kernel

$$
J_{11'}^{(2\ell-1)}(z_1,z_2|z_{1'},z_{2'}) \ + \ (11' \leftrightarrow 22'),
$$

$$
J_{11'}^{(p)} = \int_0^1 d\alpha \frac{(1-\alpha)^p}{\alpha}[\delta(z_{11'}-\alpha z_{12})-\delta(z_{11'})]\ \delta(z_{22'}).
$$

encountered in the results (3.30 - 3.34). One checks also easily that the functions $(z_1-z_2)^n$ representing the lowest weight states are eigenfunctions of this kernel. The eigenvalues are

$$
\lambda_n = \int_0^1 d\alpha \frac{(1-\alpha)^{2\ell-1+n}-1}{\alpha} = \psi(1)-\psi(2\ell+n) \tag{4.12}
$$

They are proportional to the anomalous dimensions in the parallel helicity exchange channel related to tranversity parton distributions.

The remaining kernels in the QCD operators correspond to the following particular cases of the conformal ones:

$gg \to gg$ (3.31),
$\ell_1 = \ell_2 = \ell_{1'} = \ell_{2'} = \frac{3}{2}$
$M = 0, d = -1$ results in J_{112} and $M = 0, d = -2$ results in J_{221}.
$ff \to ff$ (3.31),
$\ell_1 = \ell_2 = \ell_{1'} = \ell_{2'} = 1$
$M = 0, d = -1$ results in J_{111} and $M = 0, d = -2 + \varepsilon$ results in $J_0^{(g)}$.
$ff \to gg$ (3.34),
$\ell_1 = \ell_2 = \frac{3}{2}, \ell_{1'} = \ell_{2'} = 1, \sigma = -1$
$M = 0, d = -\frac{3}{2}$ results in $z_{12}^{-1} J_{111}$ and $M = 0, d = -\frac{5}{2} + \varepsilon$ results in $z_{12}^{-1} J_0^{(g)}$.
$gg \to ff$, (3.34)
$\ell_1 = \ell_2 = 1, \ell_{1'} = \ell_{2'} = \frac{3}{2}, \sigma = +1$
$M = 0, d = -\frac{1}{2}$ results in $z_{12} J_{112}$ and $M = 0, d = -\frac{3}{2}$ results in $z_{12} J_{221}$.
$gf \to gf$ (3.33),
$\ell_1 = \ell_{1'} = \frac{3}{2}, \ell_2 = \ell_{2'} = 1$
$M = \frac{1}{2}, d = -\frac{3}{2}$ results in $J_{111} - \frac{1}{2} J_{211}$ and $M = \frac{3}{2}, d = -\frac{3}{2} + \varepsilon$ results in $J_{11'}^{(g)} + J_{22'}^{(f)}$.
$gf \to fg$ (3.33),
$\ell_1 = \ell_{2'} = \frac{3}{2}, \ell_2 = \ell_{1'} = 1$
$M = 0, d = -\frac{3}{2}$ results in J_{121} and $M = 0, d = -\frac{1}{2} + \varepsilon$ results in $J_{11'}^{(0)}$.

5 Integrable systems of interacting parton

Consider operators acting on the tensor product $V_1 \otimes V_2 \otimes V_3$ and in particular ones acting non-trivially on two of the three spaces (as indicated by subscripts) and obeying the Yang-Baxter equation (YBE)

$$R_{12}(u-v) R_{13}(u) R_{23}(v) \;=\; R_{23}(v) R_{13}(u) R_{12}(u-v). \tag{5.1}$$

R- operators obeying this relation can be used to construct integrable quantum systems. Consider N subsystems, the quantum states of which span the representation spaces $V^{(\ell_i)}, i = 1, ..., N$ in a periodic chain-like configuration where the interaction betweeen the subsystems are due to exchanges along the chain with intermediate states out of the auxiliary space $V^{(\ell_0)}$. A complete set of commuting operators is obtained from the transfer matrix

$$t_{\ell_0}(u) = \mathrm{Sp}_0 \left(R_{01}^{(\ell_0,\ell_1)}(u) R_{02}^{(\ell_0,\ell_2)}(u) ... R_{0n}^{(\ell_0,\ell_n)}(u) \right) \tag{5.2}$$

by expansion in the spectral paramenter u. In the homogeneous case, $\ell_i = \ell, i = 1, ..., N$, these operators appear as sums of local operators (involving only a few neighbouring sites) for $\ell_0 = \ell$. The first non-trivial operator in the expasion can be considered as Hamiltonian

$$\frac{d}{du} \ln t_\ell(u)|_{u=0} = \sum H_{i,i+1}, \qquad H_{12} = P_{12} \frac{d}{du} R^{(\ell_1,\ell_2)}(u)|_{u=0} \tag{5.3}$$

Here P_{12} denotes the operator of permutation of subspaces.

The best known example of a R-operator is the one related to the Heisenberg XXX spin chain and is given by the 4×4 matrix

$$R_{12}^{(-\frac{1}{2},-\frac{1}{2})}(u) = I_{4\times 4} + \ (u + \frac{1}{2})\sigma^a \otimes \sigma^a \tag{5.4}$$

In this case the representation spaces correspond to spin $\frac{1}{2}$ ($\ell = -\frac{1}{2}$ in our notation) and are two dimensional. The tensor product with the unit matrix acting on the third subspace is not written explicitely here.

The extension by $sl(2)$ symmetry leads to another solution, called also Lax operator, appearing in YBE for one subspace of arbitrary representation ℓ and two of the fundamental one.

$$R_{12}^{(-\frac{1}{2},\ell)}(u) = I_{2\times 2} \otimes \hat{I}^{(\ell)} \ + \ (u + \frac{1}{2})\sigma^a \otimes \hat{S}_2^a = (u + \frac{1}{2})I + \hat{R}_2. \tag{5.5}$$

The goal is to construct the universal R-operator: $R^{(\ell_1,\ell_2)}$. which appears in YBE with $\ell_3 = -\frac{1}{2}$ and ℓ_1, ℓ_2 arbitrary. This particular YBE implies the defining condition conditions

$$\begin{gathered} (\hat{R}_1 + \hat{R}_2)R^{(\ell_1,\ell_2)}(u) = R^{(\ell_1,\ell_2)}(u)(\hat{R}_1 + \hat{R}_2), \\ \left(\frac{u}{2}(\hat{R}_1 - \hat{R}_2) + \hat{R}_1\hat{R}_2\right) R^{(\ell_1,\ell_2)}(u) = R^{(\ell_1,\ell_2)}(u) \left(\frac{u}{2}(\hat{R}_1 - \hat{R}_2) + \hat{R}_1\hat{R}_2\right) \end{gathered} \tag{5.6}$$

The first condition is equivalent to the condition of conformal symmetry which we solved before. The second fixes the freedom left in (3.8) for conformal operators.

We look for the universal R operator in integral form. The conditions turn into differential equations for the kernel $R(z_1, z_2|z_{1'}, z_{2'})$.

$$\begin{gathered} \left(\hat{R}_1^{(\ell_1)} + \hat{R}_2^{(\ell_2)} + \hat{R}_{1'}^{(1-\ell_1)} + \hat{R}_{2'}^{(1-\ell_2)}\right) R(z_1, z_2|z_{1'}, z_{2'}) = 0 \\ \left(\frac{u}{2}(\hat{R}_1^{(\ell_1)} - \hat{R}_2^{(\ell_2)}) + \hat{R}_1^{(\ell_1)}\hat{R}_2^{(\ell_2)} + \frac{u}{2}(\hat{R}_{1'}^{(1-\ell_1)} - \hat{R}_{2'}^{(1-\ell_2)}) + \hat{R}_{2'}^{(1-\ell_2)}\hat{R}_{1'}^{(1-\ell_1)}\right) R = 0 \end{gathered} \tag{5.7}$$

Here $\hat{R}_i^{(\ell)}$ is the non-trivial piece in the Lax operator matrix acing on z_i.

$$\hat{R}_i = \begin{pmatrix} S_i^{(\ell)0} & S_i^{(\ell)-} \\ S_i^{(\ell)+} & -S_i^{(\ell)0} \end{pmatrix} = \begin{pmatrix} 1 \\ z_i \end{pmatrix} (z_i \quad -1)\partial + \ell \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} (z_i \quad -1) + \begin{pmatrix} 1 \\ z_i \end{pmatrix} (1 \ \ 0)\right].$$

The decomposition of the 2×2 matrix structure into simple tensor products simplifies essentially the solution of (5.7). As an equivalent representation we may take the matrix element with repect of two vectors paramentrized by positions z_l and z_r.

$$\begin{gathered} < l|R_i|r >= (z_l \quad -1)\hat{R}_i \begin{pmatrix} 1 \\ z_r \end{pmatrix} = \\ -z_{li}z_{ir}\partial_i - \ell_i(z_{ir} - z_{li}) = z_{ir}^{-\ell_i+1} z_{li}^{\ell_i+1} \ \partial_i \ z_{ir}^{\ell_i} z_{li}^{-\ell_i} \end{gathered} \tag{5.8}$$

We rewite the conditions (5.7) in terms of those generic matrix elements and use the fact that some derivative terms vanish for if z_l or z_r tends to some of the points z_i. In this way one obtains that the conditions (5.7) are equivalent to four first order equations.

$$\mathcal{D}_i \ R(z_1, z_2|z_{1'}, z_{2'}) = 0. \tag{5.9}$$

Their consistency implies that the first order differential operators are similarity transformations of the simple derivatives,

$$\mathcal{D}_i = R\partial_i R^{-1},$$

and this transformation is just given by the wanted R operator kernel,

$$R(z_1, z_2|z_{1'}, z_{2'}) = z_{12}^{a_{12}} \quad z_{1'2'}^{a_{1'2'}} \quad z_{12'}^{a_{12'}} \quad z_{12'}^{a_{12'}}. \tag{5.10}$$

It is indeed a particular case of the conformal symmetric kernels with $\ell_1 = \ell_{1'}, \ell_2 = \ell_{2'}, d = u, M = -u - 1$,

$$\begin{aligned} a_{12} = u + 1 - \ell_1 - \ell_2, \quad a_{1'2'} = u - 1 + \ell_1 + \ell_2 \\ a_{12'} = -u - 1 - \ell_1 + \ell_2, \quad a_{1'2} = -u - 1 + \ell_1 - \ell_2 \\ a_{11'} = a_{22'} = \sigma = 0. \end{aligned} \tag{5.11}$$

In the special case $\ell_1 = \ell_2$ we have

$$R^{(\ell,\ell)} = \int_0^1 d\alpha_1 \int_0^{\alpha_1} d\alpha_2 \delta(z_{11'} - \alpha_1 z_{12}) \delta(z_{22'} + \alpha_2 z_{12})$$

$$(1 - \alpha_1 - \alpha_2)^{u-1+2\ell} (1 - \alpha_1)^{-u-1} (1 - \alpha_2)^{-u-1}$$

We expand for small u using (4.10)

$$\delta(z_{21'})\delta(z_{12'}) +$$

$$u \int_0^1 d\alpha \frac{(1-\alpha)^{2\ell-1}}{[\alpha]_+} \delta(z_{12'} - \alpha z_{12}) \delta(z_{21'}) \; + \; (11' \leftrightarrow 22') + \mathcal{O}(u^2)$$

Comparing with the formula for the integrable Hamiltonian (5.3) we see that the terms with $J_{11'}^{(g/f)}$ in the QCD operators (3.30 - 3.34) coincide with this Hamiltonian for $\ell_g = \frac{3}{2}$ and $\ell_f = 1$.

It is remarkable that particular cases of parton interaction operators coincide with the integrable Hamiltonian of a generalization to (non-compact) representations $\ell = 1$ or $\frac{3}{2}$ of the periodic homgeneous XXX spin chain. The multi-parton exchange contribution for parallel helicities of only gluons or only quarks is determined just by this operator. In the approximation of a large number of colours N_C closed chain configurations of gluons dominate and the corresponding contribution to the Bjorken asymptotics is calculated from the solution of the XXX integrable system. In particular the lowest energy eigenvalues determine the powers of the large scale s of this contributions.

Integrable systems corresponding to other helicity configurations of QCD parton interactions have not been constructed yet. There are some results related to open chains with three sites [18]. Besides of the terms related to the integrable Hamiltonian $J_{11'}^{(q/g)} + (1 \leftrightarrow 2)$ there are other terms in the QCD results. However there is a close relation of the other operator terms appearing in QCD to the R operator.

For the gluon case (3.31) $P_{12}R^{(\frac{3}{2},\frac{3}{2})}(u)$ for $u = -1$ results in the kernel J_{112} and for $u = -2$ in J_{221}. For the fermion case (3.32) $P_{12}R^{(1,1)}(u)$ results for $u = -1$ and -2 is J_{111} and $J_0^{(g)}$. The kernels of the annihilation interactions (3.34) are obtained as $z_{12}^{-1} P_{12} R^{(1,1)}(u)$ with $u = -1, -2$ for $ff \to gg$ and $z_{12} P_{12} R^{(\frac{3}{2},\frac{3}{2})}(u)$ with $u = -1, -2$ for $gg \to ff$.

In case of mixed quark-gluon interactions $P_{12}R^{(1,\frac{3}{2})}(u)$ with $u = -\frac{1}{2}$ reproduces J_{211} and by crossing also $J_{111} - \frac{1}{2}J_{121}$ and with $u = -\frac{3}{2}$ reproduces $J_{11'}^{(0)} = J_{11'}^{(f)} - J_{11'}^{(g)}$. In this case there is one contribution left, the kernel $J_{11'}^{(f)} + J_{11'}^{(g)}$, not reproducible from this R-operator kernel. In the supersymmetric extension this kernel appears as a component of the interaction Hamiltonian of XXX chain constructed from the superconformal $sl(2|1)$ solution of the Yang-Baxter equation [20].

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